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Valuation of Real Options Using the Minimal Entropy Martingale Measure

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Abstract

It is well known that when markets are incomplete, the no-arbitrage assumption is not sufficient to determine the exact value of an option. In this article we investigate the problem of real options valuation in multinomial trees. A concrete single real options value based on the minimal entropy martingale measure is provided. Using the MEMM to valuate options in multinomial lattices is an easy procedure which can easily be implemented by practitioners.

Keywords: Real Options Analysis, Minimal Entropy Martingale Measure, Multinomial Lattices.

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1. Introduction

The traditional approach used to valuate projects is based on the discounted cash flow method where projected future cash flows are discounted at an appropriate cost of capital. This gives the present values of the projected cash flows. For instance, let $x=(x_1,x_2,\ldots,x_N)$ be cash flows expected at the end of a one period project X. If $p_j \geq 0,\ j=1,\ldots,N$ is the probability that cash flow x_j will occur, then, the present value of the project is

$$PV(X)=\frac{p_1x_1+\ldots+p_Nx_N}{1+k} \quad (1)$$

where k is the appropriate cost of capital.

This method has been widely criticized for its failure to account for managerial flexibility in the lifetime of the project. Management of real world projects requires flexibility on the part of managers whenever they receive new information regarding progression of their projects.

It was therefore important to develop valuation models which are able to capture the value of managerial flexibility over the lifetime of the project. Real options analysis is one such methodology that has become very popular in the recent past. In real options analysis (ROA), one attempts to use the successes recorded in the valuation of financial options to the world of project management according to Klimek (2005).

There are a lot of similarities between financial and real options. For example, real options can be classified as calls or puts and their exercise style can be classified as European or American type. Moreover, the gross present value of the expected cash flows does correspond to the current value of the stock and the uncertainty in the project value corresponds to the volatility of the stock. A detailed analogue between financial and real options can be found in Dixit and Pindyck (1994), Juniper (2001), Copeland and Antikarov (2001) and many other sources.

There are also significant differences between real and financial options which prohibit the direct application of financial option valuation models such the famous option pricing formula of Black and Scholes (1973) to the area of project management. Copeland and Tufano (2004) state that real options are more complex than financial options and no one can expect to capture all the contingencies associated with them in a standard Black and Scholes option pricing formula.

Fortunately, over the years researchers have developed several other option pricing models which are more appropriate for project management. One such alternative is the binomial model of Cox et al (1979). In addition to being a perfect approximation to the Black and Scholes' option pricing formula, binomial models permit early exercises. This is very important because managers usually have options to terminate or revisit the scale of their operations prior to the original conditions for termination. Copeland and Tufano (2004) note that every node of the binomial lattice is a decision node where managers can incorporate decisions that need to be made over the life of the project. To determine the value of such a decision, you create a portfolio that replicates decision values at each node. The no-arbitrage principle ensures that the value of the replicating portfolio matches the project value. This is the basic principle underlying the binomial model of Cox et al (1979) and was widely used in the real options books of Copeland and Antikarov (2001) and Mun (2002).

The argument of replication assumes that the underlying asset is liquid and this aspect makes real options significantly different from financial options. For real options, the underlying asset is not liquid and therefore replication is not possible. One way to overcome this and still be able to use financial options theory to valuate managerial decisions in project management is to use surrogate assets. A surrogate asset or a twin security was defined by Hubalek and Schachermayer (2001) as a liquid asset whose price process is closely related to the value process of the non-liquid underlying asset of the real option.

A basic problem associated with surrogate assets is that it is practically impossible to find a financial security whose pay-offs in every state of nature match the value of the project in question.
Otherwise, the project wouldn't be a new one. Hubalek and Schachermayer (2001) show that using surrogate assets may lead to arbitrary prices which are consistent with absence of arbitrage and risk-neutral valuation. Klimek (2005) noted that this (negative) phenomenon arises from using surrogate assets whose information structure is independent from the information of the underlying it is supposed to replace. According to Klimek, a surrogate asset is appropriate only if it will match the one it replaces in every state of nature.

Copeland and Antikarov (2001) and Copeland and Tufano (2004) recommend the use of present value of the project without flexibility as the appropriate twin security. They state that the present value of the project without flexibility is the best unbiased estimate of the market value of the project. What is more correlated with the project than the project itself? This Marketed Asset Disclaimer (MAD) assumption completes the market and the replication argument of Cox et al (1979) can easily be implemented in a binomial setting. In a recent article, De Reyck et al (2006) have established that the MAD assumption is indeed plausible for real options valuation.

Klimek (2005) introduced a more general framework for valuation of real options. According to Klimek, if $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0\leq t\leq T}, P)$ is a filtered probability space with sample space $\Omega,$ probability measure P and filtration $\{\mathcal{F}\}_{0\leq t\leq T}$ with $\mathcal{F}_0=\{\Omega,\emptyset\}$ and $\mathcal{F}_T \subset \mathcal{F}$, then, a rational valuation system on this filtered probability space is defined as a family $(\pi_s^t, CF_t)_{0\leq s\leq t\leq T}$ where,

$CF_t = L^\infty(\Omega, \mathcal{F}_t, P)$ for $t = 0,\ldots,T,$ is the space of bounded cash flows.

$\pi_s^t : CF_t \longrightarrow CF_s$ for $0 \leq s \leq t \leq T$ is a linear bounded operator representing a valuation projection. These operators map non-negative functions onto non-negative functions and satisfy the following consistency conditions.

$$\pi_s^u = \pi_s^t \circ \pi_t^u \quad \text{if } 0\leq s\leq t\leq u\leq T,$$
$$\pi_t^t(CF_t) = CF_t \quad \text{for } t=0,..,T.$$

Klimek's rational valuation system is a framework consisting of several valuation models ranging from adjustment of discount factors to changing of underlying probability distributions. For instance, if $X \in CF_t$, then the present value rule can be written as

$$\pi_s^t(X) = \frac{1}{1+r_{st}} E_P[X \mid \mathcal{F}_s]$$

where r_{st} is the appropriate cost of capital for the period $[s,t]$.
The same project X can be valuated by changing the underlying probability P to a risk-neutral probability Q. The rational valuation rule can be written as

$$\pi_s^t(X) = E_Q\left[\frac{B_s X}{B_t}\middle|\mathcal{F}_s\right]$$

where B is an adapted process representing a risk free bond.

Thus, according to Klimek, the net present value rule with rightly adjusted discount factors is appropriate for project management. This can be made possible by adjusting discount factors step by step. Indeed, De Reyck et al (2006) formulated a real option valuation formula based on the present value of the project without flexibility. On the other hand, the project can be valuated by changing the underlying probability structure to a risk neutral one.

Step by step adjustment of discount factors is a time consuming procedure and may not be practically feasible, yet the same risk-neutral probabilities, once determined, can be used throughout the valuation process and this is more practical. However, the set of risk neutral probability measures is in general not a singleton and that's why the minimal entropy martingale measure is chosen as the optimal one.

The risk-neutral probability approach to valuation of real options has been restricted to binomial models. This is partly because of their simplicity but also that for higher order lattices, markets are generally incomplete and there is not a unique solution. According to Boute et al (2004), the best that has been done so far is to derive bounds for real option values and they recommend that exact solutions be investigated in further research. This article serves to provide a concrete single real options value based on the minimal entropy martingale measure. Several examples are given to illustrate the procedure for trinomial lattices but the procedure can be extended to higher order lattices.

The rest of this article is structured as follows: Section two introduces the minimal entropy martingale measure for both multiplicative and additive processes. Simple examples are then used to illustrate how this probability measure can be used to valuate real options. In Section three, more examples are given while Section four concludes.

2.0 Project Valuation Using the Minimal Entropy Martingale Measure

Consider the development of the present value of the project without flexibility in a single period. If S_0 is the current value of the project without flexibility, then, with probability $p_j > 0$, the value can jump to $a_j S_0,\ j = 1,\dots,N,\ 2 < N < \infty;\ N \in \mathbb{N}$. It is assumed that $a_i > a_j$ if $i < j$ and $a_j > 0$ for all j. Let r be the single period risk-free rate, then, the project is viable if and only if

$$a_1 > 1 + r > a_N. \qquad (3)$$

This condition is the no-arbitrage condition[2] in finance. If $1 + r > a_1$ then, no one would invest in the project because in every state of nature, the return on the project would be less than the risk-

[2] An arbitrage opportunity can be defined as the possibility to make risk-free profit in a financial market without net investment of capital.

free return. On the other hand, if $a_N > 1+r,$ then, many players would borrow at a risk free rate to invest in the project. Shortly, this opportunity would be evened.

A risk neutral probability measure can be defined as a probability measure (on a given probability space) under which the expected return on the project is equal to the risk free rate. In other words, a strictly positive probability measure $Q = (q_1, \ldots, q_N)$ defined on an appropriate probability space, is said to be a risk-neutral probability measure if and only if

$$E_Q(\frac{S_1}{S_0}) = 1+r \Leftrightarrow \sum_{i=1}^{N} a_i q_i = 1+r, \qquad (4)$$

where S_1 is the value at the end of the period. Since $N > 2,$ we are operating in an incomplete market where there are many risk-neutral probability measures. We choose a specific one that is closest to the objective probability measure in the sense that it minimizes the relative entropy with respect to the prior probability measure.

Definition 1.
Let P and Q be probability measures on a general probability space $(\Omega, \mathcal{F})$ where Ω is the sample space and $\mathcal{F}$ is a filtration representing information available in the real options market. A probability measure Q is said to be absolutely continuous with respect to P if $P(A) = 0$ implies that $Q(A) = 0$ for all $A \in \mathcal{F}.$ Notation is $Q \ll P.$ A probability measure Q is said to be equivalent to a probability measure P if and only if $P \ll Q$ and $Q \ll P.$

We define two sets $\mathfrak{M}_e$ and $\mathfrak{M}$ as follows:

$$\mathfrak{M} = \left\{ Q = (q_1, \ldots, q_N) : Q \geq 0, \sum_{i=1}^{N} q_i = 1, \sum_{i=1}^{N} q_i a_i = 1+r \right\},$$

$$\mathfrak{M}_e = \{ Q \in \mathfrak{M} : Q > 0 \}.$$

The notation $Q > 0$ implies that $q_j > 0$ for all $1 \leq j \leq N.$ Similarly $Q \geq 0$ implies that $q_j \geq 0$ for all $1 \leq j \leq N.$

The set $\mathfrak{M}$ represents martingale probability measures which are absolutely continuous with respect to the prior probability P and $\mathfrak{M}_e$ is the set of equivalent martingale measures. By the Fundamental Theorem of Asset Pricing (see Cox and Ross (1976), Harrison and Kreps (1979), Harrison and Pliska (1981, 1983) and Schachermayer (2004)), the set $\mathfrak{M}_e \neq \emptyset,$ since we have assumed a viable (arbitrage-free) market for the real options market. In the current setting however, one can deduce from the following proposition adopted from Klimek (2005) that the no arbitrage condition in (3) ensures that $\mathfrak{M}_e$ (equivalently $\mathfrak{M}$) is non-empty.

Proposition
Let $c_1, c_2, \ldots, c_n$ be n (ordered) real numbers such that c_1 is the smallest and c_n is the largest and let $c \in \mathbb{R}.$ There exists a strictly positive probability measure $P = (p_1, \ldots, p_n)$ such that $\sum_{k=1}^{n} p_k c_k = c$ if and only if $c_1 < c < c_n.$

Definition 2. Relative Entropy
Let Q and P be probability measures on a finite probability space $(\Omega, \mathcal{F}).$ The relative entropy of Q with respect to a probability measure P is defined as a number

$$I(Q,P) = \sum_{\omega \in \Omega} Q(\omega) \ln \frac{Q(\omega)}{P(\omega)}.$$

We understand throughout the paper that $0\ln(0) = 0.$

Basic properties of relative entropy are well known. For example,

$$0 \le I(Q,P) \le \infty.$$

Relative entropy gives a measure of how different two probability distributions are. It is not a metric though.

Definition 2. (MEMM)

The probability measure $\hat{Q} \in \mathfrak{M}$ is called the minimal entropy martingale measure (MEMM) if it satisfies

$$I(\hat{Q},P) = \min_{Q \in \mathfrak{M}} I(Q,P).$$

For a single period-finite probability model, the MEMM can be determined using the method of Lagrangian multipliers. The basic optimization problem is given as follows:.

$$\begin{cases} \min \sum_{i=1}^{N} q_i \ln(\frac{q_i}{p_i}) \\ \text{s.t.} \quad \sum q_i a_i = 1 + r \quad \sum_{i=1}^{N} q_i = 1 \text{ and } q_i \ge 0 \text{ for all } 1 \le i \le N. \end{cases} \tag{5}$$

Problem (5) has a unique solution $\hat{Q} = (\hat{q}_1, \ldots, \hat{q}_N)$ given by

$$\hat{q}_i = \frac{p_i e^{-\gamma a_i}}{\sum_{i=1}^{N} p_i e^{-\gamma a_i}}, \quad i = 1, 2, \ldots N \tag{6}$$

provided that there exists a constant γ which satisfies the following equation:

$$\sum_{i=1}^{N} p_i a_i e^{-\gamma a_i} = (1+r) \sum_{i=1}^{N} p_i e^{-\gamma a_i}. \tag{7}$$

The following lemma due to Frittelli (1995) links the existence and uniqueness of γ to the no arbitrage (NA) assumption in equation (3).

Lemma.

There are no arbitrage opportunities if and only if equation (7) has a unique solution.

Proof: See Appendix.

Let $\tilde{x}_j, j = 1, \ldots, N$ be the state j payoff of the project $\widetilde{X}$ with a real option and let $\bar{\pi}$ be the value from immediate exercise. Then, the minimal entropy value of the project with a real option is given as

$$\pi(\widetilde{X}) = \max \left\{ \frac{1}{1+r} \sum_{j=1}^{N} \tilde{x}_j \hat{q}_j, \tilde{\pi} \right\}$$

and the minimal entropy value of the real option is therefore given as

$$\pi(\widetilde{X}) - PV(X).$$

The following example is taken from De Reyck et al (2006).

Example 1. Abandonment Option
An abandonment option can be defined as an option to close out an investment prior to the fulfillment of the original conditions for termination.

Consider a project X with three possible outcomes, \$1.2, \$ 1.0 or \$ 0.8 and respective probabilities 0.25, 0.5 and 0.25. The risk-free interest rate is 5% and the cost of capital is 10%. Using these values, the present value of the project (without flexibility) is found to be \$ 0.9091. Determine the value of an abandonment option with a payoff of one dollar exercisable only at the end of the period.

Solution:

A risk-neutral valuation argument would derive bounds on the value $\pi(\tilde{X})$ of the project with options as follows:

$$\inf_{Q\in\mathfrak{M}} E_Q\left[\frac{\widetilde{X}}{1+r}\right] \le \pi(\widetilde{X}) \le \sup_{Q\in\mathfrak{M}} E_Q\left[\frac{\widetilde{X}}{1+r}\right]$$

where

$$\mathfrak{M} = \left\{ Q = \left(q,\ \frac{17}{22} - 2q,\ q + \frac{5}{22} \right) : 0 < q < \frac{17}{44} \right\}.$$

Thus, the value of the abandonment option ranges between $\$\ 0.043$ and $\$\ 0.117$. With no any other criterion, the best that can be done is to determine the range in which option values lie With this information, the MEMM was found to be $\widehat{Q} = (0.149,\ 0.474,\ 0.377)$ and the minimal entropy value of the abandonment option was found to be $\$\ 0.072$. The certainty-equivalent version[3] of the net present value formula proposed by De Reyck et al (2006) gives an option value between $\$\ 0.063$ and $\$\ 0.076$. As illustrated in the next example, the two approaches (certainty-equivalence and risk-neutral valuation) can yield a range of option values which are non-overlapping.

The minimal entropy martingale measure is not the only martingale probability measure that can be used to derive an exact solution. Other martingale probability measures such as the minimal variance martingale measure can be used.
Let $\mu = \sum_{i=1}^{N} p_i a_i$ and $\sigma^2 = \sum_{i=1}^{N} p_i (a_i - \mu)^2$. The minimal variance martingale measure (MVMM) (as given in Frittelli (1995) is defined as $\widetilde{Q} = (\tilde{q}_1, \ldots, \tilde{q}_N)$ where

$$\tilde{q}_j = p_j \left(1 + \frac{\mu - r}{\sigma^2} (\mu - a_j) \right),\ j = 1, \ldots, N.$$

If valued using the MVMM, the value of the abandonment option in Example 1 is found to be $\$\ 0.069$.

Remark 1.
For purposes of pricing options, the MEMM is more appropriate than the minimal variance martingale measure because as seen in equation (6), it is always equivalent to the objective probability measure. On the other hand, the minimal variance martingale measure is not in general equivalent to the objective probability measure (see a counterexample in Frittelli (1995)).

[3] Using average market returns of 12% with standard deviation of 20% as in De Reyck (2006).

Example 2.
Consider an option to contract or shrink a project. This can be achieved by selling or subletting part of the production facilities to another company. When exercised at a strike price K, the project's present value is shrunk by a factor β. The single period minimal entropy value of the project with the option to contract is given as

$$\pi(\widetilde{X}) = \frac{1}{1+r}\sum_{j=1}^{N}\hat{q}_j \max(x_j, \beta x_j + K),$$

if the option is exercisable only at the end of the period. Otherwise, this value must be compared with the value from immediate exercise and the maximum of the two will be the option value. In Example 1, suppose that the project can be contracted by 25% thereby saving $\$\ 0.28$ in operating expenses. What is the value of the contraction option?

Solution:
A risk-neutral approach gives an option value between $ 0.039 and $ 0.047. The certainty-equivalent version of the net present value formula proposed by De Reyck et al (2006) gives an option value between $ 0.074$ and $0.104. Using the minimal entropy martingale measure, the value of the option was found to be $0.042. In other words, the total minimal entropy value of the project with a contraction option was found to be $ 0.951. As this example shows, certainty equivalence and risk neutral approaches can lead to no-overlapping interval of prices.

2.1 The MEMM when the Underlying Process is Additive.

Copeland and Antikarov (2001) illustrate some examples for valuation of real options when the underlying process follows an additive stochastic process. In this respect, a brief discussion of the MEMM when the underlying process is additive is made.

We assume as before that the market consists of a risky security and a risk free bond and there are N possible states where $N > 2$.
The price processes of the two securities are given as follows:

$$B_0 = 1, B_1 = R = 1 + r;$$

$$S_0 = S\ ,\ S_1 = \begin{cases} S + b_1 & \textit{with probability } p_1 \\ S + b_2 & \textit{with probability } p_2 \\ \vdots & \vdots \\ S + b_N & \textit{with probability } p_N \end{cases}$$

where r is equal to a one-period deterministic interest rate, where for $i = 1, 2, \ldots, N;\ b_i \in \mathbb{R}_+$ and $p_i > 0$. We interpret S as the present value of the project without flexibility and b_j as the incremental value of the project without flexibility when the state of the world is j. It is also assumed that t $b_i > b_j; \forall\, i < j$. In addition, the no arbitrage condition is given by

$$b_1 > sr > b_N. \qquad (8)$$

A strictly positive probability measure $Q = (q_1, \ldots, q_N)$ is said to be a martingale measure if and only if

$$E_Q\left(\frac{S_1}{1+r}\right) = S_0 \Leftrightarrow \sum_{i=1}^{N} b_i q_i = rS.$$

So, the single period problem is

$$
\begin{cases}
\min \sum_{i=1}^{N} q_i \ln(\frac{q_i}{p_i}) \\
\text{s.t.} \sum q_i b_i = sr, \ \sum_{i=1}^{N} q_i = 1 \text{ and } q_i \geq 0 \ \forall 1 \leq i \leq N.
\end{cases} \qquad (9)
$$

By the method of Lagrangian multipliers, (9) has a unique solution $\widehat{Q} = (\hat{q}_1, \ \ldots, \ \hat{q}_N)$ given by

$$\hat{q}_i = cp_i e^{-\gamma b_i}, \quad i = 1, 2, \ldots N$$

where $c = \sum_{j=1}^{N} p_j e^{-\gamma b_j}$ and γ is a scalar which satisfies the following equation:

$$\sum_{i=1}^{N} p_i b_i e^{-\gamma b_i} = rs \sum_{i=1}^{N} p_i e^{-\gamma b_i}$$

Existence and uniqueness of γ is guarantied by the no arbitrage condition in (10).

3.0 More Examples on the Valuation of Real Options Using the MEMM

3.1 Compound Options

The following example is adopted (with permission) from Copeland and Tufano (2004).

Example

Copano, a chemical firm is considering a phased investment plant. It will cost \$ 60 million immediately for permits and preparations, which will take a year. At the end of year one, the firm can invest \$ 400 million to complete the design phase. Managers believe that once the design phase is over, the firm has a two year window during which it can make a final investment worth \$ 800 million needed to build the plant.

This is an example of what is commonly known as compound options or sequential options. A \$ 60 million investment now creates the right but not the obligation to invest \$ 400 million in one year, which if exercised creates the right but not the obligation to invest \$ 800 million to purchase the plant.

Based on NPV calculations, the firm is assumed to be worth \$ 1000 million today but future values are uncertain and are random in nature. The volatility of the project is assumed to be 18.23% per annum and the risk-free rate is assumed to be 8% per annum.

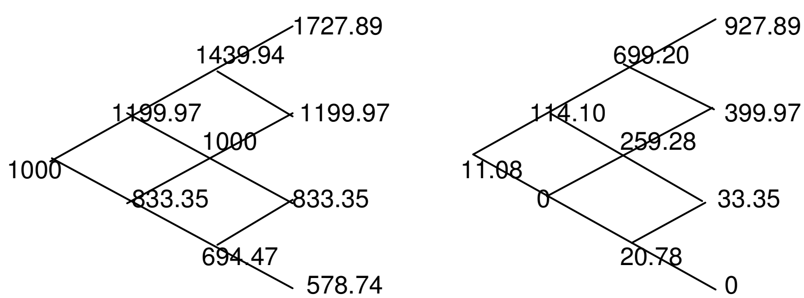
At the discount rate of 10.83% if the firm decides to invest in the second year, the present value of costs will be \$ 1072.2 million but if it decides to invest in the third year, the present value of the costs will be \$ 1008.56 million. In both cases, net present value rule shows that it is worthless investing in this project.

However, investments at the end of year one, two or three are options and will be exercised if deemed worth. Using real options analysis, we are able to determine the value of such options available in the future lifetime of the project. The procedure is illustrated using binomial and trinomial lattices. Instead of constructing a replicating portfolio as did Copeland and Tufano (2004), a risk-neutral approach approach is used. The downside of the former is that it is restricted to binomial models. The risk-neutral approach as described in this article can be extended to higher-order lattices.

Determining the value of the project using binomial models

We use the Marketed Asset Disclaimer assumption of Copeland and Antikarov (2001). A binomial event tree for the value of the project without options is shown in Figure 1 (a). Let S be the value of the project at the beginning of a single period. With risk-neutral probability $q > 0$, the value at the end of the period can be $uS,\ u>0$ or with risk-neutral probability $1-q$, it can be $dS,\ d>0$ where q,u and d are parameters in the binomial model[4] of Cox et al (1979). Having constructed a binomial lattice for the value of the project with no options, then familiar single step risk neutral pricing relations are used to determine the value of a compound option. The procedure is summarized in the following four steps:

- At the end of the third year, the value of the project with a real option at node k is $C_3^k = \max\left\{S_3^k - 800, 0\right\},\ k = 1,\ldots,4.$
- At the end of the second year, the value of the project with a real option at node k is $$C_2^k = \max\left\{S_2^k - 800, \frac{1}{1+r}(qC_3^k + (1-q)C_3^{k+1})\right\},\ k = 1,2,3.$$
- At the end of the first year, the value of the project with a real option at node k is $$C_1^k = \max\left\{0, \frac{1}{1+r}(qC_2^k + (1-q)C_2^{k+1}) - 400\right\},\ k = 1,2.$$
- The time zero value of the project with a real option is $$C_0 = \max\left\{0, \frac{1}{1+r}(qC_2^1 + (1-q)C_2^2) - 60\right\}.$$



(a) Binomial Event Tree. (b) Option Values

Figure 1: Binomial models showing the development of underlying project and option values.

Thus, the value of the project was found to be $ 11.08 million. This is also the value of the sequential options since the net present value approach gives a zero value. Option values at every node of the binomial tree are shown in Figure 1 (b).

Binomial models are easy to implement, but they lack some pedagogical appeal. If the value can go up or down, it can as well be assumed to stay the same. For binomial models, risk neutral valuation and method of replication yield the same value. For higher order lattices, perfect replication is not possible. Moreover, there are many equivalent martingale measures and we focus on the one which minimizes relative entropy.

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4 The CRR model is generally understood to imply up probability $q = \frac{e^{r\Delta t}-d}{u-d}$ and jump amplitudes $u = \frac{1}{d} = \exp(\sigma\Delta t)$ where σ is the volatility in annualized terms. With the same jump amplitudes , they also derived an alternative up probability $q = \frac{1}{2}\left(1 + \frac{(r-\frac{\sigma^2}{2})\sqrt{\Delta t}}{\sigma}\right)$ which is the Taylor series expansion of the former. With this alternative parameterization, the value of the project is found to be $ 11.74 Million.

Determining the value of the project using Trinomial models

To compute the MEMM value of the project using trinomial models, some more assumptions in addition to the MAD assumption of Copeland and Antikarov (2001) are needed. In particular, we assume that value S of the project without flexibility can jump to uS with probability p_1, to S with probability p_2 or to dS with probability p_3. The corresponding jump amplitudes are assumed to be $u = \exp(\sigma\Delta t)$ and $d = \exp(-\sigma\Delta t)$.

Equation (1) is used to estimate the probabilities. That is,

$$\frac{p_1 uS + p_2 S + p_3 dS}{1+k} = S.$$

Together with the condition that these probabilities add to one, p_1 and p_3 are computed as follows:

$$p_1 = \frac{(1+k-e^{-\sigma}) - p_2(1-e^{-\sigma})}{e^{\sigma} - e^{-\sigma}}$$

and

$$p_3 = 1 - p_2 - p_1,$$

where k is the appropriate cost of capital and the middle jump probability $0 \le p_2 \le 1$ is chosen in such a way that $0 \le p_1, p_3 \le 1$. With these parameters, the MEMM can be derived from equations (8) and (9).

A trinomial event tree which is similar to Figure 1 but with a middle jump for the project without flexibility was developed. Using familiar steps as in the binomial model, option values were determined for different values of p_2. Results are displayed in Table 4.

Table 4: MEMM value (in million of dollars) of Copano.

p_2	**Project Value**
0.00	11.08
0.05	8.41
0.10	5.71
0.15	2.95
0.20	0.12
$\geq$ 0.25	0.00

Observation

As $p_2 \to 0$, the minimal entropy value approaches the value computed using binomial models.

3.2 Expansion Option

Suppose that a firm wants to expand project X by a factor e. If K is the cost of expansion or the exercise price of the option, then, the single period minimal entropy value of the expansion option is written as

$$\pi(\widetilde{X}) = \max\left\{\frac{1}{1+r}\sum_{j=1}^{N}\hat{q}_j \max(x_j,\ ex_j - K),\ \bar{\pi}\right\}$$

where $\bar{\pi}$ is the value from immediate exercise.

Example
The present value of the future profitability of a growth firm[5] is found to be $ 400 million. The volatility of the logarithmic returns on the projected cash flows was estimated to be 35% per annum and the risk-free rate was assumed to be 7% per annum. The firm has an option to expand and double its operations by acquiring its competitor for a sum of $ 250 million at any time over the next five years. What is the value of the expansion option?

Solution
A trinomial lattice for modeling the growth potential of the firm can be constructed using the following parameters. $a_1 = e^{0.35\sqrt{\Delta t}}, a_2 = 1, a_3 = e^{-0.35\sqrt{\Delta t}}$ where $\Delta t = \frac{T}{M}, T = 5$ and M is the number of time steps. In addition, it is assumed that the objective probabilities are given as $p_1 = p_3 = \frac{1}{2}(1 - p_2)$ and $p_2 \geq 0.$ Using familiar single step risk neutral pricing relations, we can determine the value of the expansion option. In Table 1, option values are reported for different values of p_2 and number of time steps M.

Table 1: MEMM value (in million of dollars) of an option to expand

	M			
p_2	**5**	**50**	**100**	**500**
0.0	86.77	88.62	88.65	88.86
0.1	85.23	86.51	86.55	86.63
0.2	83.19	84.29	84.33	84.42
0.3	80.98	82.11	82.15	82.25
0.4	78.78	80.01	80.05	80.15
0.5	76.71	78.03	78.08	78.18
0.6	74.84	76.26	76.32	76.42
$\geq$0.7	0.00	0.00	0.00	0.00

Observation
The value of the firm with an option to expand decreases for increasing p_2 and settles at a value of $ 550 million which is the firm's static net present value without flexibility. The value of the expansion option can be found by subtracting this value from the total value of the firm with an expansion option. Note that for $p_2 = 0,$ option values using the MEMM are very close to those given by the binomial models as computed in Mun (2002). Any observable differences are purely computational.

Remark 2
The computation of the minimal entropy martingale measure depends on the parameters a_1, a_2, a_3 together with the corresponding probabilities $p_1, p_2, p_3.$ In other words; it incorporates the modeler's subject beliefs about the possible outcome of the underlying project. In general, a different set of parameters might lead to different conclusions.

For example, as in Yamada and Primbs (2004), suppose that the jump amplitudes are $a_1 = \exp(\mu\Delta t + \alpha\sqrt{\Delta t}),\ a_2 = \exp(\mu\Delta t)$ and $a_3 = \exp(\mu\Delta t - \alpha\sqrt{\Delta t})$ where μ is the average growth rate of the firm as measured by the mean of the logarithmic returns on the projected cash

5 This example was extracted from p. 175 of Mun's Real Option Analysis Book.

flows and is assumed to be 10% per annum, and $\alpha = \frac{\sigma}{\sqrt{1 - p_2}}$, $0 \le p_2 < 1$. In addition, suppose that $p_1 = p_3 = \frac{1}{2}(1 - p_2)$.

Then, the minimal entropy values of an option to expand the project are as shown in Table 2.

Table 2: MEMM value (in million of dollars) of an option to expand

	M			
p_2	**5**	**50**	**100**	**500**
0.0	85.43	88.72	88.72	88.82
0.1	86.53	88.62	88.73	88.84
0.2	86.66	88.58	88.70	88.83
0.3	86.31	88.61	88.74	88.84
0.4	85.47	88.57	88.74	88.84
0.5	84.47	88.54	88.71	88.84
0.6	86.14	88.62	88.68	88.84
$\ge$0.7	86.66	88.49	88.48	88.82

In this case, the value of the firm with an expansion option is always above its static net present value, no matter the value of middle jump probability.

So far, the procedure for derivation of the minimal entropy martingale measure involves a constrained optimization of relative entropy with the necessary constraint that the expected return of the project without flexibility be equal to the risk-free gross return. This way, the minimal entropy martingale measure correctly values the project without flexibility and is closest in the entropic distance to the prior probability measure. More generally, the price process of any other marketed security (which is relevant to the pricing of the current project in question) can be used in the constraint equation. The chosen security could be the present value of the project without flexibility, the price process of an exchange traded asset or it could be the present value process of another project which shares similar features with the project in question. For example, if a company is interested in valuating a new gold mine, the value process of another gold mine with similar features can be used.

Formally, suppose that we want to determine the value of a contingent claim $g(Y_T)$ written on a non-tradeable stochastic underlying process Y, which is defined on some probability space $(\Omega, \mathcal{F}, P)$ where the terms have their usual meaning. In addition, there is an actively traded marketed asset whose discounted value at time $t \le T$ is denoted by S_t^* and is assumed to be correlated with Y. The market comprised of the risky marketed asset, the risk free bond and the contingent asset (real option) is assumed to be arbitrage free. The problem at hand is to determine the arbitrage free price of such a contingent claim. The first step is to find a minimizer for the following problem.

$$\begin{cases} \min \; I(Q,P) \\ s.t. \; E_Q[S_t^* \mid F_s] = S_s^*, \; 0 \le s \le t \le T, \; and \; E_P[\frac{dQ}{dP}] = 1. \end{cases}$$

The minimizer $\widehat{Q}$ exists by the no arbitrage assumption. It correctly prices the marketed asset and is the closest to P among all probability measures having finite relative entropy. Indeed, one can include as many independent constraints as desired if the interest is to derive the probability measure which correctly prices all these benchmark securities and is closest to the prior in the entropic distance. This procedure is commonly known as marking to market or model calibration according to Kruk (2004).

The time t minimal entropy price of the real option with pay off $g(Y_T)$ is therefore given by

$$\pi(g(Y_T)) = E_{\hat{Q}}\left[\frac{B_t}{B_T} g(Y_T) \mid \mathcal{F}_t\right].$$

4.0 Conclusion and recommendations

The minimal entropy martingale measure was used to solve the problem of real options valuation in multinomial lattices. The MEMM yields a concrete single option value which is in some sense optimal. As illustrated by practical examples, the procedure is easy to implement and can be adopted by practitioners. Unlike the replication argument in binomial models, the procedure can easily be extended to higher order lattices. The minimal entropy martingale measure takes into account objective probabilities as well as jump amplitudes. It therefore incorporates modelers' subjective beliefs about the expected outcomes of the project with no flexibility.

Empirical research is necessary to determine how close minimal entropy prices are to actual values. It was also shown that two approaches; certainty-equivalence and risk-neutral valuation can yield a range of option values which are non-overlapping. The relationship between these two approaches needs to be investigated in further research.

Lastly, I have discussed that with the current approach, the price process of any other marketed security (which is relevant to the pricing of the current project in question) can be used in the constraint equation. The chosen security could be the present value of the project without flexibility, the price process of an exchange traded asset or it could be the present value process of another project which shares similar features with the project in question. A forth coming article will discuss this issue more formally and in detail.

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References

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Political Economy*. 81(3): 637-654.

Boute, R., Demeulemeester, E. and Herroelen, W. (2004). A real options approach to project management. *International Journal of Production Research*. 42(9): 1715-1725.

Copeland, T. and Antikarov, V. (2001). *Real Options.* New York: Texere LLC.

Copeland, T. and Tufano, P. (2004). A real-world way to manage real options. *Harvard Business Review*. 82(3): 90-99.

Cox, J.C., Ross, S. and Rubinstein, M. (1979). Option pricing: A simplified approach. *Financial Economics.* 7(3): 229-263.

De Reyck, B., Degraeve , Z. and Vandenborre, R. (2006). Project options valuation with net present value and decision tree analysis. *European Journal of Operations Research*, doi: 10.1016/j.jeor.2006.07.047.

Dixit, A. K. and Pindyck, R, S. (1994). *Investment under Uncertainty*. New Jersey: Princeton University Press.

Frittelli, M. (1995). Minimal entropy criterion for pricing in one period incomplete markets. *Working Paper. University of Brescia, Italy.*

Harrison, M.J. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Economic Theory*. 20(3): 381-408.

Harrison, M.J. and Pliska, S.R, (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*. 11(3): 215-260.

Harrison, M.J. and Pliska, S.R, (1983). A stochastic calculus model of continuous trading: complete markets. *Stochastic Processes and their Applications*. 15(3): 313-316.

Hubalek, F. and Schachermayer, W. (2001). The limitations of no-arbitrage arguments for real options. *International Journal of Theoretical and Applied Finance*. 4(2): 361-373.

Juniper, J. (2001). Extending real options theory to account for property investment under conditions of uncertainty and incomplete markets. *Seventh Annual Pacific-Rim Real Estate Society Conference Adelaide South Australia.*

Klimek, M. (2005). Rational Valuation Systems and Compatibility of Information. *Working Paper, Uppsala University Sweden.*

Kruk, L. (2004). Limiting distributions for minimum relative entropy calibration. *Applied Probability*. 41(1): 35-50.

Mun, J. (2002). *Real Options Analysis*. John Wiley and Sons Inc.

Appendix

Proof of Lemma 1

We give a slightly modified proof of Frittelli's Lemma.

Let $M(x) = \sum_{i=1}^{N} a_i q_i(x)$ where $q_j(x) = \dfrac{p_i e^{-xa_i}}{\sum_{j=1}^{N} p_j e^{-xa_j}}$, $j = 1, \ldots, N$. Clearly, $M \in C^1(\mathbb{R})$ and is the mean of a random variable $\mathbf{a}$ whose probability distribution is given as

$$P(\mathbf{a} = a_j) = q_j(x) = \frac{p_i e^{-xa_i}}{\sum_{j=1}^{N} p_j e^{-xa_j}}, i = 1, 2, \ldots N.$$

To show that there exists a unique γ such that $M(\gamma) = 1 + r$, we write the function M as follows:

$$M(x) = \frac{p_1 a_1 e^{-a_1 x} + p_2 a_2 e^{-a_2 x} + \ldots + p_N a_N e^{-a_N x}}{p_1 e^{-a_1 x} + p_2 e^{-a_2 x} + \ldots + p_N e^{-a_N x}}.$$

Then, $\lim_{x \to \infty} M(x) = a_N < 1 + r < a_1 = \lim_{x \to -\infty} M(x).$

Therefore, by the Intermediate Value Theorem, we conclude that there exists a constant $\gamma \in \mathbb{R}$ such that $M(\gamma) = 1 + r$.
To show uniqueness, it is sufficient to show that $M'(x) < 0$ for all $x \in \mathbb{R}$.

$$M'(x) = \sum_{i=1}^{N} a_i \frac{dq_i(x)}{dx} = -\sum_{i=1}^{N} a_i^2 q_i(x) + \left(\sum_{i=1}^{N} a_i q_i(x) \right)^2 = -Var(\mathbf{a}) < 0.$$

For the converse, suppose that there exists a constant γ satisfying equation (7). Let

$$q_j^* = \frac{p_j e^{-\gamma a_j}}{\sum_{i=1}^{N} p_i e^{-\gamma a_i}}, \; j = 1, 2, \ldots, N.$$

Then, $Q^* = (q_1^*, q_2^*, \ldots, q_N^*)$ is an equivalent probability measure on the given sample space. It is also a martingale probability measure since

$$\sum_{j=1}^{N} a_j S_0 q_j^* = \frac{\sum_{j=1}^{N} p_j a_j e^{-\gamma a_j}}{\sum_{j=1}^{N} p_j e^{-\gamma a_j}} = (1+r) S_0.$$

The conclusion follows from the Proposition.
As a corollary, if $N = 2$, then $\hat{Q} = (\hat{q}_1, \hat{q}_2)$ where $\hat{q}_1 = \frac{1+r-d}{u-d}$ and $\hat{q}_2 = \frac{u-1-r}{u-d}$ which is the unique equivalent martingale measure for binomial models.

Let C_u be the option value in the upstate and C_d be the option value in the down state, in a one step binomial model, then, the option value c at the beginning of the period is given by the following risk neutral valuation formula.

$$c = \frac{1}{1+r} \left(q_1 C_u + q_2 C_d \right) = \frac{1}{1+r} \left(\frac{C_u (1+r-d)}{u-d} + \frac{C_d (u-1-r)}{u-d} \right). \tag{10}$$

On the other hand, suppose that a replicating portfolio can be constructed with x units of the risky asset (whose value today is S) and y units of the risk free asset (whose value today is one). With u and d being the up and down factors, the replicating portfolio must satisfying the following two equations.

$$\begin{aligned} xuS + y(1+r) &= C_u \\ xdS + y(1+r) &= C_d. \end{aligned} \tag{11}$$

The solution to (11) is given by $x = \frac{c_u - c_d}{(u-d)S}$ and $m = \frac{uC_d - dC_u}{(1+r)(u-d)}$.

By the no arbitrage condition, the value of the option must equal to the value of the replicating portfolio. In other words,

$$c = xS + m = \frac{1}{1+r} \left(\frac{C_u (1+r-d)}{u-d} + \frac{C_d (u-1-r)}{u-d} \right). \tag{12}$$

This is the same as in (11). The advantage of using the risk neutral approach as opposed to replication is that the same risk-neutral probabilities can be used throughout the valuation process, yet a replicating portfolio has to be constructed for every valuation step in a multi-period binomial lattice.